ON (2, 3) COMPOUND INVOLUTIONS.

By Temple Rice Hollcroft.

1. *Introduction.*—This paper together with a preceding one* is intended to complete the discussion and classification of (2, 3) point correspondences between two planes.

In the general point correspondence a point and its successive images do not form a closed group. In a compound involution, however, the correspondence closes up at the second application of the transformation. Choosing (x) and (x') respectively as the double and triple planes, to a point P' of (x') correspond three points P_1, P_2, P_3 of (x) such that to each of these image points correspond the original point P' and a residual point P'_0 of (x'). To P'_0 as well as to P' correspond the three points P_1, P_2, P_3 of (x). The point correspondence thus established is involutorial. The essential difference between this correspondence and the general (2, 3) point correspondence is that now the residual images coincide.

The first known paper discussing point correspondences with both planes multiple was published in 1889.† This is a very simple (2, 2) compound involution in which the lines of either plane correspond to conics of the other. Burali-Forti‡ later obtained certain (2, 2) compound involutions by combining two (1, 2) involutions and showed that the case treated by Visalli is included in these. Finally the (2, 2) compound involutions have been classified and six independent types obtained by F. R. Sharpe and Virgil Snyder.§

2. *Outline of Method.*—Suppose that a (1, 2) point correspondence has been established between the planes (x') and (y'). Then to a point P' of (y') correspond two points P'_1, P'_2 of (x') and to either P'_1 or P'_2 of (x') corresponds the one point P' of (y'). Also to the lines through P'_1 or P'_2 correspond rational curves of (y') all passing through P'.

Assume further that a (1, 3) point correspondence exists between the planes (x) and (y). Then to a point P of (y) correspond three points

* T. R. Hollcroft: "A Classification of General (2, 3) Point Correspondences between two Planes," Am. Jour. of Math., Vol. XLI (1919), pp. 5–24.

† P. Visalli, "La trasformazione quadratica (2, 2)," *Rend. del Circ. Mat. di Palermo*, Vol. 3 (1889), pp. 165–170.

‡ "Sulle trasformazione (2, 2) che si possono ottenere mediante due trasformazioni doppie," *Rend. del Circ. Mat. di Palermo*, Vol. 5 (1891), pp. 91–99.

§ "Types of (2, 2) Point Correspondences between two Planes," *Trans. Amer. Math. Soc.*, Vol. XVIII (1917), pp. 409–414.

P_1, P_2, P_3 of (x), to each of the image points P_1, P_2, P_3 corresponds P of (y) and to the lines through P_1, P_2 or P_3 correspond rational curves of (y) all passing through P.

Now consider the planes (y) and (y'). Both contain nets of rational curves through the points P and P' respectively. The planes are therefore birationally equivalent, that is, a (1, 1) correspondence exists between them such that P corresponds to P' and P' to P and such that curves in (y) which are images of the lines of (x) and curves in (y') which are images of the lines of (x') are transformed reciprocally one into the other.

Then to two points P'_1, P'_2 of $(x)'$ corresponds P' of (y') to which corresponds P of (y) to which correspond P_1, P_2, P_3 of (x). Reciprocally, P_1, P_2, P_3 of (x) go into P of (y), thence into P' of (y'), thence into P'_1, P'_2 of (x'). The planes (x) and (x') are therefore so related that to two points of (x') correspond three points of (x) and to these three image points correspond the original two of (x'), that is, a (2, 3) compound involution has been established between them.

There are other methods of establishing (2, 3) compound involutions, but it will be proved later that all the independent types of (2, 3) compound involutions can be obtained by the method outlined above.

3. *General Properties.*—The curves of either plane are transformed into curves of the other by three transformations, two rational and one irrational. Since the two planes (y) and (y') are birationally equivalent, they may be considered the same plane in which the basis points of the two systems are rationally separable. Then the image in (x') of a curve of (x) is obtained by applying to its image in (y), considered as being also in (y') and retaining its basis points as fixed points of (y'), the transformation from (y') to (x'). Thus if a line of (x) goes into $C_n(y)$ with basis points kP_i* and a line of (y') corresponds to $C'_m(x')$ with basis points lP'_j, then the line of (x) corresponds to $C'_{mn}(x')$ with basis points lP'_{jn} and $2kP'_i$. A similar series of transformations relates (x') to (x).

If the two images P'_1, P'_2 of a point P of (x) coincide, P is on $L(x)$ the branch-point curve of (x). The locus of the corresponding coincidences is $K'(x')$, the coincidence curve of (x'). $K'(x')$ is the jacobian of the net of image curves of the lines of (y'). Its complete image is $L(x)$. The image of $L(x)$ is $K'(x')$ counted six times.

The coincidence curve $K(x)$ is the jacobian of the net of image curves in (x). The residual curve $\Gamma(x)$ is the co-jacobian of this net. To a point P' on $L'(x')$ corresponds two coincident points on $K(x)$ and one on $\Gamma(x)$. The complete image of either $K(x)$ or $\Gamma(x)$ is $L'(x')$. The image of $L'(x')$ is $K(x)$ counted four times and $\Gamma(x)$ counted twice. The branch-point

* Hollcroft, loc. cit., page 7.

curves and their corresponding coincidence or residual curves are not in (1, 1) correspondence as in the case of $(1, n)$ or general (2, 3) point correspondences.

The non-basic intersections of $K(x)$ and $\Gamma(x)$ are all contacts to each of which corresponds two cusps on $L'(x')$. The images of both the cusps of $L'(x')$ coincide at the point of tangency of $K(x)$ and $\Gamma(x)$. There are thus only a finite number of points of (x') whose three images in (x) coincide, namely, the cusps of $L'(x')$.

To a point of $K'(x')$ correspond three points of $L(x)$ to each of which corresponds the original point on $K'(x')$ counted twice. To a point of $K(x)$ correspond two points of $L'(x')$ to each of which correspond the original point of $K(x)$ counted twice and a point of $\Gamma(x)$, whose images are those same two points of $L'(x')$.

The non-basic intersections of $K'(x')$ with $L'(x')$, of $L(x)$ with $K(x)$ and of $L(x)$ with $\Gamma(x)$ are equal in number and are all contacts. To each tangency of $K'(x')$ with $L'(x')$ corresponds a tangency of $L(x)$ with $K(x)$ and of $L(x)$ with $\Gamma(x)$. To each of these two points corresponds the same point of contact of $L'(x')$ and $K'(x')$. Since $K'(x')$ is counted six times as the image of $L(x)$, each contact of $L'(x')$ with $K'(x')$ counts as six contacts. To four of these correspond the four coincident contacts of $(K)^4$ and L and to two of them, the two coincident contacts of $(\Gamma)^2$ and L.

4. *Types of* (1, 2) *Involutions.*—There are three independent types of (1, 2) point correspondences.* Type 1, the Geiser type, is obtained by the intersections of any line with an associated conic of a net, or by the cubics of a net through seven basis points. Type 2, the Jonquières type, is given by the intersections of a line of the pencil P with a curve of order n of a net having an $(n - 2)$-fold point at P. Type 3, the Bertini type, is given by the variable intersections of a cubic of a pencil with an associated sextic having double points at eight of the basis points of the pencil.

5. *Types of* (1, 3) *Involutions.*—Five independent types of (1, 3) point correspondences have been recently found.† Type 1 is given by the intersections of a plane field of lines with the associated cubics of a net; Type 2 by the intersections of a line pencil and a net of curves of order n with $(n - 3)$-fold points at the vertex of the pencil; Type 3 by the intersections of two conics of a system of conics through one basis point; Type 4 by the intersections of two cubics of a net of cubics through six basis points; Type 5 by the intersections of a cubic of a pencil with an associated curve of a system of curves of order nine having triple points at eight of the nine basis points of the pencil of cubics.

* See Pascal's (German) "Repertorium der höheren Mathematik," second edition, Vol. 2, pp. 366–370.

† Anna Mayme Howe, "A Classification of Plane Involutions of Order Three," AM. JOUR. OF MATH., Vol. XLI (1919), pp. 25–40.

6. *Types of* (2, 3) *Compound Involutions.*—Fifteen types of (2, 3) compound involutions are obtained by combining the types of (1, 2) and (1, 3) involutions. For convenience these have been divided into three classes according to the (1, 2) involution employed. There are five types in each class, namely, the five (1, 3) involutions combined with the (1, 2) involution determining that class. The following table shows the combination and type number for each type. The symbol C_n; jP_i denotes a curve of order n with j basis points each of multiplicity i. The arabic numerals are the type numbers. The type is established by combining the systems in the row and column in which the type number is found.

Class	Double Plane / Triple Plane	C_1; C_3	C_1, P_1; C_m, P_{m-3}	C_2, P_1; C_2, P_1	$C_3, 6P_1$; $C_3, 6P_1$	$C_3, 8P_1$; $C_9, 8P_3$
I	C_1; C_2	1	2	3	4	5
II	C_1, P_1; C_n, P_{n-2}	6	7	8	9	10
III	$C_3, 8P_1$; $C_6, 8P_2$	11	12	13	14	15

CLASS I.

7. *Type* 1.—In the (1, 2) point correspondence, denote the double plane by (y') and the simple plane by (x'). The defining equations of the Geiser type used in Class I may then be written,

$$(1)\quad \sum_{i=1}^{3} x'_i y'_i = 0,$$

$$(2)\quad \sum_{i=1}^{3} y'_i v'_i(x') = 0,$$

where $v'_i(x')$ are general conics of (x').

(The following notation will be used in describing the curve systems of the planes being discussed:

The symbol " $\sim$ " meaning " corresponds to ";

L, L', K, K', Γ, fixed curves as heretofore defined;

P, Q, P', Q', basis points;

$\bar{P}$, $\bar{P}'$ variable points;

p, genus of curve being described;

Subscripts of curves denote their order;

Subscripts of points denote their multiplicity.)

$C'_1(y') \sim C'_3(x')$; $p = 1$, $7P'_1$.

$C'_1(x') \sim C'_3(y')$; $p = 0$, $\bar{P}'_2$.

$K'_6(x')$; $p = 3$, $7P'_2$. $\quad L'_4(y')$; $p = 3$.

In the (1, 3) point correspondence, denote the triple plane by (y) and the simple plane by (x). The defining equations of Type 1 are:

$$(1)\quad \sum_{i=1}^{3} x_i y_i = 0,$$

(2) $\sum_{i=1}^{3} y_i u_i(x) = 0,$

where $u_i(x)$ are general cubics of (x).

$C_1(y) \sim C_4(x)$; $p = .3$, $13P_1$.

$C_1(x) \sim C_4(y)$; $p = 0$, $\bar{P}_3$.

$K_9(x)$; $p = 15$, $13P_2$.

$\Gamma_{22}(x)$; $p = 15$, $13P_6$.

$L_{10}(y)$; $p = 15$, 21 cusps.

8. *Image Curves.*—In the (2, 3) compound involution established between the planes (x) and (x') by the two preceding involutions, to obtain the image in (x') of $C_1(x)$ we use the transformation from (y') to (x') on $C_4(y)$. $C_1(x) \sim C'_{12}(x') \equiv C_4(x'_2v'_3 - x'_3v'_2, x'_3v'_1 - x'_1v'_3, x'_1v'_2 - x'_2v'_1) = 0$. $C'_{12}(x')$ has $7P'_4$ given by $v'_1/x'_1 = v'_2/x'_2 = v'_3/x'_3$ and two variable triple points which are images of $\bar{P}_3$ of $C_4(y)$. C'_{12} is of genus 7.

The image of $C'_1(x)'$ is found by using the transformation from (y) to (x) on $C'_3(y')$. $C'_1(x') \sim C_{12}(x) \equiv C'_3(x_2u_3 - x_3u_2, x_3u_1 - x_1u_3, x_1u_2 - x_2u_1) = 0$. $C_{12}(x)$ has $13P_3$ given by $u_1/x_1 = u_2/x_2 = u_3/x_3$ and three variable double points, images of $\bar{P}'_2$ of $C'_3(y')$. C_{12} is of genus 13.

$L(x)$ is obtained by applying the transformation from (y) to (x) on $L'_4(y')$. This gives $L_{16}(x)$ of genus 27 with $13P_4$. The image of $L_{16}(x)$ in (y) is a curve of order 12 which is birationally equivalent to $L'_4(y')$ counted three times. The image of $L_{16}(x)$ is, therefore, $K'_6(x')$ counted six times. $K'_6(x')$ has $7P'_2$ and is of genus 3. The complete image of $K'_6(x')$ is $L_{16}(x)$.

$L'(x')$ is obtained by applying the transformation from (y') to (x') on $L_{10}(y)$. This gives $L'_{30}(x')$ with $7P'_{10}$, 42 cusps and of genus 49. The image of $L'_{30}(x')$ in (y') is a curve of order 20 which is birationally equivalent to $L_{10}(y)$ counted twice. Then the image of $L'_{30}(x')$ in (x) is $K_9(x)$ counted four times and $\Gamma_{22}(x)$ counted twice. $K_9(x)$ has $13P_2$ and is of genus 15. $\Gamma_{22}(x)$ has $13P_6$ and is of genus 15. The complete image of either $K_9(x)$ or $\Gamma_{22}(x)$ is $L'_{30}(x')$.

$K_9(x)$ and $\Gamma_{22}(x)$ have 21 tangencies to each of which correspond two cusps on $L'_{30}(x')$. Each of these two cusps corresponds to that same tangency counted as two points on K_9 and one on Γ_{22}.

Each pair, $K_9(x)$ and $L_{16}(x)$, $\Gamma_{22}(x)$ and $L_{16}(x)$, $K'_6(x')$ and $L'_{30}(x')$ has 20 contacts. The points of tangency of $K'_6(x')$ and $L'_{30}(x')$ correspond to those of $K_9(x)$ with $L_{16}(x)$ and of $\Gamma_{22}(x)$ with $L_{16}(x)$ and reciprocally.

9. *Successive Images of Lines.*—The image of $C'_1(x')$ as found is $C_{12}(x)$ with $13P_3$ and $3\bar{P}_2$. To $C_{12}(x)$ corresponds $C_9(y)$, birationally equivalent to $[C'_3(y')]^3$. The image of $C'_3(y')$ is $C_1(x)$ and a residual $C'_8(x')$ with $7P'_3$. Then to $C_{12}(x)$ corresponds $(C'_1)^3(C'_8)^3$ in (x'). The complete image of $C'_8(x')$ is $C_{12}(x)$. C'_8 and K'_6 have 6 nonbasic intersections at the intersections

of C'_1 and K'_6. C'_8 and L'_{30} have 30 non-basic intersections distinct from the intersections of C'_1 and L'_{30}. C_{12} and L_{16} in (x) have 18 points of contact. These are the images of the 6 intersections of C'_1 or C'_8 with K'_6. Reciprocally, the images of these 18 tangencies are the 36 intersections of C'_1 or C'_8 with $(K'_6)^6$, three tangencies corresponding to each set of 6 coincident intersections. To the 30 intersections of C'_1 with L'_{30} and to the 30 intersections of C'_8 with L'_{30} correspond the 30 intersections of C_{12} with K_9, also of C_{12} with Γ_{22} and reciprocally.

The image of $C_1(x)$ is $C'_{12}(x')$ with $7P'_4$ and $2\bar{P}'_3$. To C'_{12} corresponds $C'_8(y')$ which is birationally equivalent to $[C_4(y)]^2$. The image of $C_4(y)$ is $C_1(x)$ and a residual C_{15} with $13P_4$. Then $C'_{12}(x')$ corresponds to $(C_1)^2(C_{15})^2$ in (x). The complete image of $C_{15}(x)$ is $(C'_{12})^2$. C_{15} and L_{16} have 32 intersections which together with the 16 intersections of C_1 and L_{16} correspond to the 16 intersections of C'_{12} with K'_6 and reciprocally. The curves C_{15} and K_9 have 31 and C_{15} and Γ_{22} have 18 intersections. C'_{12} and L'_{30} have 80 intersections to which correspond the 40 intersections of C_1 and C_{15} with K_9 and the 40 intersections of C_1 and C_{15} with Γ_{22}. The 80 intersections of C'_{12} and L'_{30} correspond to the 160 intersections of C_1 and C_{15} with $(K_9)^4$ and the 80 intersections of C_1 and C_{15} with $(\Gamma_{22})^2$.

The images of two lines $C'_1(x')$, $\bar{C}'_1(x')$ are $C_{12}(x)$, $\bar{C}_{12}(x)$ respectively, which have 27 non-basic intersections. The images of C_{12} and $\bar{C}_{12}$ are $(C'_1)^3(C'_8)^3$ and $(\bar{C}'_1)^3(\bar{C}'_8)^3$ respectively. C'_8 and $\bar{C}'_8$ intersect in one non-basic point to which correspond three intersections of C_{12} and $\bar{C}_{12}$. To each of these three intersections correspond the intersection of C'_8 with $\bar{C}'_8$ and the intersection of C'_1 with $\bar{C}'_1$. These three intersections of C_{12} and $\bar{C}_{12}$ are the images of the intersection of the two lines of (x'). For the remaining 24 intersections of C_{12} and $\bar{C}_{12}$, to each of the three points in each set of eight corresponds a point of intersection of C'_1 with $\bar{C}'_\infty$ and also an intersection of $\bar{C}'_1$ with C'_8. To these latter correspond respectively the 24 intersections of C_{12} with $\bar{C}_{12}$.

The preceding paragraph, with the exception of the order of the image curves of (x), applies equally well to all five types of Class I.

The images of two lines $C_1(x)$, $\bar{C}_1(x)$ are $C'_{12}(x')$, $\bar{C}'_{12}(x')$ respectively, intersecting in 32 points. The images of C'_{12} and $\bar{C}'_{12}$ are $(C_1)^2(C_{15})^2$ and $(\bar{C}_1)^2(\bar{C}_{15})^2$ respectively. C_{15} and $\bar{C}_{15}$ intersect in 17 non-basic points. To the intersection of C_1, $\bar{C}_1$ and two of the intersections of C_{15}, $\bar{C}_{15}$ correspond two intersections of C'_{12}, $\bar{C}'_{12}$ and reciprocally. To the 15 intersections of C_1, $\bar{C}_{15}$, the 15 intersections of $\bar{C}_1$, C_{15} and the remaining 15 intersections of C_{15}, $\bar{C}_{15}$ correspond the remaining 30 intersections of C'_{12}, $\bar{C}'_{12}$ and reciprocally. This paragraph, with the exception of the order of the image curves of (x'), applies equally well to types 6 and 11.

Since for all the types the details and methods are similar, only the results will be given for the remaining ones.

10. *Type* 2.—Type 2 of the (1, 3) point correspondences has the defining equations,

(1) $x_1y_1 + x_2y_2 = 0$,

(2) $y_1\varphi_1(x) + y_2\varphi_2(x) + y_3\varphi_3(x) = 0$,

where $\varphi_i(x)$ is a curve of order m with an $(m - 3)$-fold point at the vertex of the line pencil, $Q \equiv (0, 0, 1)$. The basis point of (y) is $Q \equiv (0, 0, 1)$.

$C_1(y) \sim C_{m+1}(x)$, $p = 2m - 3$, Q_{m-2}, $(6m - 6)P_1$.

$C_1(x) \sim C_{m+1}(y)$, $p = 0$, Q_m.

$K_{2m}(x)$; $p = 6m - 9$; Q_{2m-4}, $(6m - 6)P_1$.

$\Gamma_{4m-2}(x)$; $p = 6m - 9$; Q_{4m-6}, $(6m - 6)P_2$.

$L_{4m-2}(y)$; $p = 6m - 9$; Q_{4m-6}, $(6m - 6)$ cusps.

For the (2, 3) compound involution:

$C_1'(x') \sim C_{3m+3}(x)$; $p = 6m - 5$; Q_{3m-6}, $(6m - 6)P_2$, $3\overline{P}_2$.

$C_1(x) \sim C_{3m+3}'(x')$; $p = 2m + 1$; $7P_{m+1}'$; $2P_m'$.

$L_{4m+4}(x)$; $p = 8m + 3$; Q_{4m-8}, $(6m - 6)P_4$.

$K_{2m}(x)$; $p = 6m - 9$; Q_{2m-4}, $(6m - 6)P_1$.

$\Gamma_{4m-2}(x)$; $p = 6m - 9$; Q_{4m-6}, $(6m - 6)P_2$.

$L_{12m-6}'(x')$; $p = 20m - 23$; $7P_{4m-2}'$, $2P_{4m-6}'$, $(12m - 12)$ cusps.

$K_6'(x')$; $p = 3$; $7P_2'$.

11. *Type* 3.—Type 3 of the (1, 3) involutions has the defining equations:

(1) $\sum_{i=1}^{3} y_iu_i(x) = 0$,

(2) $\sum_{i=1}^{3} y_iv_i(x) = 0$.

wherein $u_i(x)$ and $v_i(x)$ are conics through the basis point Q.

$C_1(y) \sim C_4(x)$; $p = 2$; Q_2, $9P_1$.

$C_1(x) \sim C_4(y)$; $p = 0$; $\overline{P}_3$.

$K_9(x)$; $p = 9$; Q_5, $9P_2$.

$\Gamma_{14}(x)$; $p = 9$; Q_6, $9P_4$.

$L_8(y)$; $p = 9$; 12 cusps.

For the (2, 3) compound involution:

$C_1'(x') \sim C_{12}(x)$; $p = 10$; Q_6, $9P_3$, $3\overline{P}_2$.

$C_1(x) \sim C_{12}'(x')$; $p = 7$; $7P_4'$, $2\overline{P}_3$.

$L_{16}(x)$; $p = 23$; Q_8, $9P_4$.

$K_9(x)$; $p = 9$; Q_5, $9P_2$.

$\Gamma_{14}(x)$; $p = 9$; Q_6, $9P_4$.

$L_{24}'(x')$; $p = 33$; $7P_8'$, 24 cusps.

$K_6'(x')$; $p = 3$; $7P_2'$.

12. *Type* 4.—Type 4 of the (1, 3) point correspondences has defining

equations of the same form as those of type 3, the $u_i(x)$ and $v_i(x)$ now representing cubics through $6P_1$.

$C_1(y) \sim C_6(x)$; $p = 4$; $6P_2$, $9P_1$.

$C_1(x) \sim C_6(y)$; $p = 0$; $\overline{P}_5$.

$K_{15}(x)$; $p = 22$; $6P_5$, $9P_2$.

$\Gamma_{42}(x)$; $p = 22$; $6P_{14}$, $9P_8$.

$L_{12}(y)$; $p = 22$; 33 cusps.

For the (2, 3) compound involution:

$C'_1(x') \sim C_{18}(x)$; $p = 16$; $6P_6$, $9P_3$, $3\overline{P}_2$.

$C_1(x) \sim C'_{18}(x')$; $p = 11$; $7P'_6$, $2P'_5$.

$L_{24}(x)$; $p = 45$; $6P_8$, $9P_4$.

$K_{15}(x)$; $p = 22$; $6P_5$, $9P_2$.

$\Gamma_{42}(x)$; $p = 22$; $6P_{14}$, $9P_8$.

$L'_{36}(x')$; $p = 67$; $7P'_{12}$, 66 cusps.

$K'_6(x')$; $p = 3$; $7P'_2$.

13. *Type* 5.—Type 5 of the (1, 3) point correspondences has defining equations of the same form as those of type 3, wherein the $u_i(x)$ are cubics with nine basis points ($8P_1$ and Q_1) and the $v_i(x)$ are curves of order nine with $8P_3$.

$C_1(y) \sim C_{12}(x)$; $p = 7$; $8P_4$, Q_1, $12P_1$.

$C_1(x) \sim C_{12}(y)$; $p = 0$; P_9, $19\overline{P}_2$.

$K_{24}(x)$; $p = 28$; $8P_8$, Q_2, $12P_1$.

$\Gamma_{60}(x)$; $p = 28$; $8P_{20}$, Q_{14}, $12P_4$.

$L_{18}(y)$; $p = 28$; P_{12}, 42 cusps.

For the (2, 3) compound involution:

$C'_1(x') \sim C_{36}(x)$; $p = 25$; $8P_{12}$, Q_3, $12P_3$, $3\overline{P}_2$.

$C_1(x) \sim C'_{36}(x')$; $p = 23$; $7P'_{12}$, $2P'_9$, $38\overline{P}_2$.

$L_{48}(x)$; $p = 43$; $8P_{16}$, Q_4, $12P_4$.

$K_{24}(x)$; $p = 28$; $8P_8$, Q_2, $12P_1$.

$\Gamma_{60}(x)$; $p = 28$; $8P_{20}$, Q_{14}, $12P_4$.

$L'_{54}(x')$; $p = 91$; $7P'_{18}$, $2P'_{12}$, 84 cusps.

$K'_6(x')$; $p = 3$; $7P'_2$.

CLASS II.

14. In establishing the types of (2, 3) compound involutions of Class II the Jonquières (1, 2) point correspondence is used to relate the planes (x') and (y'). Its defining equations are:

(1) $y'_1 x'_1 + y'_3 x'_3 = 0$,

(2) $y'_1 \psi'_1(x') + y'_2 \psi'_2(x') + y'_3 \psi'_3(x') = 0$,

where $\psi'_i(x)'$ is a curve of order n with an $(n-2)$-fold point at $Q' \equiv (0, 1, 0)$, the vertex of the line pencil (1).

$C'_1(y') \sim C'_{n+1}(x')$; $p = n - 1$; Q'_{n-1}, $(4n-2)P'_1$.

$C_1'(x') \sim C_{n+1}'(y')$; $p = 0$; Q_n'.

$L_{2n}'(y')$; $p = 2n - 2$; Q_{2n-2}'.

$K_{2n}'(x')$; $p = 2n - 2$; Q_{2n-2}', $(4n - 2)P_1'$. This is the coincidence curve of (x') for the five types of compound involutions in this class.

The coincidence and residual curves of (x) for each type of compound involution are described in the (1, 3) involution employed for that type. The (1, 3) point correspondences used in type 6, 7, 8, 9, 10 are described in paragraphs 7, 10, 11, 12, 13 respectively.

15. *Type* 6.—

$C_1'(x') \sim C_{4n+4}(x)$; $p = 5n + 3$; $13P_{n+1}$, $3P_n$.

$C_1(x) \sim C_{4n+4}'(x')$; $p = 4n - 1$; Q_{4n-4}', $(4n - 2)P_4'$, $2\overline{P}_3'$.

$L_{8n}(x)$; $p = 16n - 8$; $13P_{2n}$, $3P_{2n-2}$.

$L_{10n+10}'(x')$; $p = 10n + 29$; Q_{10n-10}', $(4n - 2)P_{10}'$, 42 cusps.

16. *Type* 7.—

$C_1' \sim C_{(m+1)(n+1)}$; $p = (2m-1)(n+1)-2$; $Q_{(m-2)(n+1)}$, $(6m-6)P_{n+1}$, $3P_n$.

$C_1 \sim C_{(m+1)(n+1)}'$; $p = mn + n - 1$; $Q_{(m+1)(n-1)}'$, $(4n - 2)P_{m+1}'$, $2P_m'$.

$L_{2n(m+1)}$; $p = 4(mn + n - 2)$; $Q_{2n(m-2)}$, $(6m - 6)P_{2n}$, $3P_{2n-2}$.

$L_{(4m-2)(n+1)}'$; $p = 4mn + 12m - 2n - 19$; $Q_{(4m-2)(n-1)}'$, $(4n - 2)P_{4m-2}'$, $2P_{4m-6}'$, $12m - 12$ cusps.

17. *Type* 8.—

$C_1' \sim C_{4n+4}$; $p = 4n + 2$; Q_{2n+2}, $9P_{n+1}$, $3P_n$.

$C_1 \sim C_{4n+4}'$; $p = 4n - 1$; Q_{4n-4}', $(4n - 2)P_4'$, $2\overline{P}_3'$.

L_{8n}; $p = 14n - 8$; Q_{4n}, $9P_{2n}$, $3P_{2n-2}$.

L_{8n+8}'; $p = 8n + 17$; Q_{8n-8}', $(4n - 2)P_8'$, 24 cusps.

18. *Type* 9.—

$C_1' \sim C_{6n+6}$; $p = 6n + 4$; $6P_{2n+2}$, $9P_{n+1}$, $3P_n$.

$C_1 \sim C_{6n+6}'$; $p = 6n - 1$; Q_{6n-6}', $(4n - 2)P_6'$, $2\overline{P}_5'$.

L_{12n}; $p = 18n - 8$; $6P_{4n}$, $9P_{2n}$, $3P_{2n-2}$.

L_{12n+12}'; $p = 12n + 43$; Q_{12n-12}', $(4n - 2)P_{12}'$, 66 cusps.

19. *Type* 10.—

$C_1' \sim C_{12n+12}$; $p = 9m + 7$; $8P_{4n+4}$, Q_{n+1}, $12P_{n+1}$, $3P_n$.

$C_1 \sim C_{12n+12}'$; $p = 12n - 1$; Q_{12n-12}', $(4n - 2)P_{12}'$, $2P_9'$, $38\overline{P}_2'$.

L_{24n}; $p = 24n - 8$; $8P_{8n}$, Q_{2n}, $12P_{2n}$, $3P_{2n-2}$.

L_{18n+18}'; $p = 18n + 55$; Q_{18n-18}', $(4n - 2)P_{18}'$, $2P_{12}'$, 84 cusps.

CLASS III.

20. In establishing the types of (2, 3) compound involutions of this class, the Bertini (1, 2) point correspondence is used to relate the planes (x') and (y'). Its defining equations are,

(1) $y_1'u_1'(x') + y_2'u_2'(x') = 0$,

(2) $\sum_{i=1}^{3} y_i'v_4'(x') = 0$,

wherein the $u'_i(x')$ are cubics with eight basis points $(8P'_i)$ and the $v'_i(x')$ are curves of order six with $8P_2$.

$C'_1(y') \sim C'_6(x')$; $p = 2$; $8P'_2$, $2P'_1$.

$C'_1(x') \sim C'_6(y')$; $p = 0$; 2 consecutive P'_3, $4\bar{P}'_2$.

$L'_6(y')$; $p = 4$; 2 consecutive P'_3, the basis line (image of P'_1 of (x')) being the common 6-fold tangent to L'_6 and to each $C'_6(y')$.

$K'_9(x')$; $p = 4$; $8P'_3$. This is the coincidence curve of (x') for the five types of compound involutions in this class.

The (1, 3) point correspondences used in types 11, 12, 13, 14, 15 are described in paragraphs 7, 10, 11, 12, 13 respectively.

21. *Type* 11.—

$C'_1 \sim C_{24}$; $p = 28$; $13P_6$, 6 consecutive P_3, $12\bar{P}_2$.

$C_1 \sim C'_{24}$; $p = 11$; $8P'_8$, $2P'_4$, $2P'_3$.

L_{24}; $p = 40$; $13P_6$, 6 consecutive P_3.

L'_{60}; $p = 59$; $8P'_{20}$, $2P'_{10}$, 42 cusps.

22. *Type* 12.—

$C'_1 \sim C_{6m+6}$; $p = 12m - 8$; Q_{6m-12}, $(6m - 6)P_6$, 6 consecutive P_3, $12\bar{P}_2$.

$C_1 \sim C'_{6m+6}$; $p = 3m - 2$; $8P'_{2m+2}$, $2P'_{m+1}$, $2P'_m$.

L_{6m+6}; $p = 12m + 4$; Q_{6m-12}, $(6m - 6)P_6$, 6 consecutive P_3.

L'_{24m-12}; $p = 24m - 25$; $8P'_{8m-4}$, $2P'_{4m-2}$, $2P'_{4m-6}$, $12m - 12$ cusps.

23. *Type* 13.—

$C'_1 \sim C_{24}$; $p = 22$; Q_{12}, $9P_6$, 6 consecutive P_3, $12\bar{P}_2$.

$C_1 \sim C'_{24}$; $p = 11$; $8P'$, $2P_4$, $2\bar{P}_3$.

L_{24}; $p = 34$; Q_{12}, $9P_6$, 6 consecutive P_3.

L'_{48}; $p = 41$; $8P'_{16}$, $2P'_8$, 24 cusps.

24. *Type* 14.—

$C'_1 \sim C_{36}$; $p = 34$; $6P_{12}$, $9P_6$, 6 consecutive P_3, $12\bar{P}_2$.

$C_1 \sim C'_{36}$; $p = 17$; $8P'_{12}$, $2P'_6$, $2\bar{P}'_5$.

L_{36}; $p = 46$; $6P_{12}$, $9P_6$, 6 consecutive P_3.

L'_{72}; $p = 79$; $8P'_{24}$, $2P'_{12}$, 66 cusps.

25. *Type* 15.—

$C'_1 \sim C_{72}$; $p = 52$; $8P_{24}$, Q_6, $12P_6$, 6 consecutive P_3, $12\bar{P}_2$.

$C_1 \sim C'_{72}$; $p = 35$; $8P'_{24}$, $2P'_{12}$, $2P'_9$, $38\bar{P}'_2$.

L_{72}; $p = 64$; $8P_{24}$, Q_6, $12P_6$, 6 consecutive P_3.

L'_{108}; $p = 109$; $8P'_{36}$, $2P'_{18}$, $2P'_{12}$, 84 cusps.

26. *Conditions on Curve Systems.*—We shall now find the necessary and sufficient conditions on the curve systems that a (2, 3) point correspondence be a compound involution.

In a (2, 3) compound involution any P_i of (x) determines the other two P_i and either P'_1 or P'_2 of (x') determines P'_2 or P'_1 respectively. Then the two image points of (x') are in a simple involution and any image point of

(x) is in a simple involution with the other two. Then we can map the plane (x) on a triple plane (y) by a $(1, 3)$ transformation and the plane (x') on a double plane (y') by a $(1, 2)$ transformation such that the two planes (y) and (y') are in $(1, 1)$ correspondence. For a $(1, 3)$ point correspondence can always be mapped on a triple plane by equations of the form,

(1) $\varphi_1(x)/y_1 = \varphi_2(x)/y_2 = \varphi_3(x)/y_3$,

where $\varphi_i(x) = 0$ define a net of curves with three variable intersections. The plane (y) is mapped on the plane (y') by the Cremona transformation,

(2) $ky'_i = f_i(y)$, $i = 1, 2, 3$.

and thence on (x') by the transformation,

(3) $\psi'_1(x')/y'_1 = \psi'_2(x')/y'_2 = \psi'_3(x')/y'_3$,

wherein $\psi'_i(x') = 0$ defines a net of curves in (x') with two variable intersections. By means of (2) eliminate y_i and y'_i from (1) and (3) and we have the relation,

(4) $F_1(x)/F'_1(x') = F_2(x)/F'_2(x') = F_3(x)/F'_3(x')$

wherein any two F_i have three variable intersections and any two F'_i have two variable intersections. Since (4) is the necessary and sufficient condition that the curve system in either plane be a net, we have deduced the theorem:

The necessary and sufficient condition that a (2, 3) *point correspondence be a compound involution is that the image curves in either plane form a net.*

It may be here noted that when a pair of defining equations for a $(2, 3)$ point correspondence have equations of the Bertini type locating the two images in the triple plane, the curve system of that plane forms a net and the point correspondence is always a compound involution.

27. Proof will now be given for the theorem:

The sufficient condition that a (2, 3) *point correspondence be a compound involution is that in either plane both components of the curve system defining the image points form pencils.*

Let the correspondence be defined by

(1) $u_1(x)u'_1(x') + u_2(x)u'_2(x') = 0$,

(2) $v_1(x)v'_1(x') + v_2(x)v'_2(x') = 0$,

such that the components of the curve system of (x) $[(x')]$ intersect in three [two] points. If both components of either system form a pencil both components of the other system also form a pencil because only two homogeneous parameters remain in each equation.

Choose any point (x'_1) of (x') and consider it as fixed. Then (x'_1) determines two curves of the system in (x') which intersect in another fixed point (x'_2). To (x'_1) correspond three fixed points (x_1), (x_2), (x_3) of (x) which lie at the intersection of two of the curves of the system in (x). Any one of the image points of (x) uniquely determines the other two, because their defining curves form pencils.

Since (x_1), (x_2), (x_3) are images of the fixed point (x_1'), the following relations hold:

(1) $u_1(x_1)/u_2(x_1) = u_1'(x_1')/u_2'(x_1')$,
$v_1(x_1)/v_2(x_1) = v_1'(x_1')/v_2'(x_1')$,
(2) $u_1(x_2)/u_2(x_2) = u_1'(x_1')/u_2'(x_1')$,
$v_1(x_2)/v_2(x_2) = v_1'(x_1')/v_2'(x_1')$,
(3) $u_1(x_3)/u_2(x_3) = u_1'(x_1')/u_2'(x_1')$,
$v_1(x_3)/v_2(x_3) = v_1'(x_1')/v_2'(x_1')$.

These relations give the relation in (x):

(I) $u_1(x_1)/u_2(x_1) = u_1(x_2)/u_2(x_2) = u_1(x_3)/u_2(x_3)$,
$v_1(x_1)/v_2(x_1) = v_1(x_2)/v_2(x_2) = v_1(x_3)/v_2(x_3)$.

The relation (I) means that any one of the three image points of (x) determines the other two—a fact already known. Likewise in (x') since the components form pencils and either image point determines the other, this relation must hold:

(I′) $u_1'(x_1')/u_2'(x_1') = u_1'(x_2')/u_2'(x_2')$,
$v_1'(x_1')/v_2'(x_1') = v_1'(x_2')/v_2'(x_2')$.

We know that each of the three points of (x) correspond to (x_1') and a residual point of (x'). We wish now to prove that the residual point is (x_2') for each of the three image points of (x). To do this we must prove the following relations:

(4) $u_1(x_1)/u_2(x_1) = u_1'(x_2')/u_2'(x_2')$,
$v_1(x_1)/v_2(x_1) = v_1'(x_2')/v_2'(x_2')$,
(5) $u_1(x_2)/u_2(x_2) = u_1'(x_2')/u_2'(x_2')$,
$v_1(x_2)/v_2(x_2) = v_1'(x_2')/v_2'(x_2')$,
(6) $u_1(x_3)/u_2(x_3) = u_1'(x_2')/u_2'(x_2')$,
$v_1(x_3)/v_2(x_3) = v_1'(x_2')/v_2'(x_2')$.

Relations (4), (5) and (6) are shown to be true by (I′) and (1), (2), (3) respectively. Then (x_2') is the residual image of each of the three points (x_1), (x_2), (x_3). Also from (4), (5) and (6) the three images of (x_2') are (x_1), (x_2), (x_3). The point correspondence is therefore a compound involution.

28. *Pencil Cases.*—In accordance with the foregoing theorem, the twelve independent types of general (2, 3) point correspondences* when the equations have but two homogeneous parameters reduce to particular forms of compound involutions. It will be interesting to see to what types of compound involutions they are reduced. (In the following, the Roman numerals refer to types of general (2, 3) point correspondences, the arabic to types of (2, 3) compound involutions.)

The pencil form of Type I is a special case of Type VII for $m = 3$, $n = 2$, which is a particular form of Type 7.

* Hollcroft, loc. cit., page 9.

The pencil form of Type II is a special case of the alternate way of writing Type VII and is therefore a particular case of Type 7.

The pencil form of Type III reduces by quadric inversion to the pencil form of Type I or Type II, depending on the choice of the triangle of inversion, either of which is a special case of Type VII and therefore a particular form of Type 7.

The pencil form of Type IV is a special case of the pencil form of Type VIII for $n = 2$.

The pencil form of Type V goes into the pencil form of Type III by quadric inversion, and is a particular form of Type 7.

The pencil form of Type VI is a special case of Type VII.

The pencil form of Type VII may have its defining equations written as in Type VII each with one less term, or in the alternate form:

(1) $x_1\psi_1' + x_2\psi_2' = 0$

(2) $x_1'\varphi_1 + x_2'\varphi_2 = 0.$

In either case it is a particular form of Type 7.

The pencil form of Type VIII is a particular form of Type 9.

The pencil form of Type IX is a particular form of Type 10.

The pencil form of Type X is transformed into the alternate form of Type VII by quadric inversion and is a particular form of Type 7.

The pencil form of Type XI reduces to a special case of the pencil form of Type III by quadric inversion and is therefore a particular form of Type 7.

The pencil form of Type XII reduces by quadric inversion to a special case of the pencil form of Type IV and is therefore a particular form of Type 9.

29. *Cyclic Cases.*—In Types II, IV and a particular case of Type V of (1, 3) point correspondences, the three image points in the simple plane constitute a cyclic projectivity of period three.* This property of the image points is retained in the (2, 3) compound involutions evolved from these types, for a point of (x') corresponds to one point of (y') thence to one point of (y) and thence to three points of (x). So the three images of a point of (x') are also images of the corresponding point of (y).

Therefore the second, fourth and a particular form of the fifth types of all three classes of (2, 3) compound involutions are such that the three image points in (x) constitute a cyclic projectivity of period three. Furthermore in those types of all three classes, two lines of (x') determine respectively 9, $2n + 1$ and 18 triads of points in (x) each triad constituting a cyclic projectivity of period three.

In all types of (1, 2) point correspondences the two image points of the simple plane are in a simple involution. This same property holds for the

* A. M. Howe, loc. cit., pp. 39–49.

two image points in the triple plane for all (2, 3) compound involutions. Two lines of (x) determine the following numbers of pairs of points in (x'), each pair constituting a cyclic projectivity of period two:

Types 1, 6, 11	16 pairs
Types 2, 7, 12	$2m+1$ pairs
Types 3, 8, 13	16 pairs
Types 4, 9, 14	36 pairs
Types 5, 10, 15	63 pairs.

30. *Completeness of the Classification.*—It has been shown that the necessary and sufficient condition that a (2, 3) point correspondence be a compound involution is that the image curves of either plane form a net. When this is true the double and triple planes may always be mapped on two other planes by (1, 3) and (1, 2) point correspondences respectively and those two other planes are birationally equivalent. Therefore any (2, 3) compound involution may be obtained by combining a (1, 3) and a (1, 2) involution.

It has been proved that all (1, 2) involutions may be reduced to one of the three independent types herein described.*

By the same method it has been shown that all (1, 3) involutions may be reduced to one of the five independent types described.†

Since all (2, 3) compound involutions can be obtained by combining (1, 2) and (1, 3) involutions, there can be no more independent types of (2, 3) compound involutions than there are possible combinations of the independent types of (1, 2) and (1, 3) involutions. Also since all the (1, 2) and (1, 3) involutions used in these combinations are independent of each other, each combination gives an independent type of (2, 3) compound involutions. There are, then, fifteen independent types of (2, 3) compound involutions and the classification is complete.

WELLS COLLEGE,
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* E. Bertini: "Recherche sulle trasformazioni univoche involutorie nel piano," *Annali di Matematica*, Ser. 8, Vol. 8 (1877), pp. 244–286.

† A. M. Howe, loc. cit., pp. 38–39.